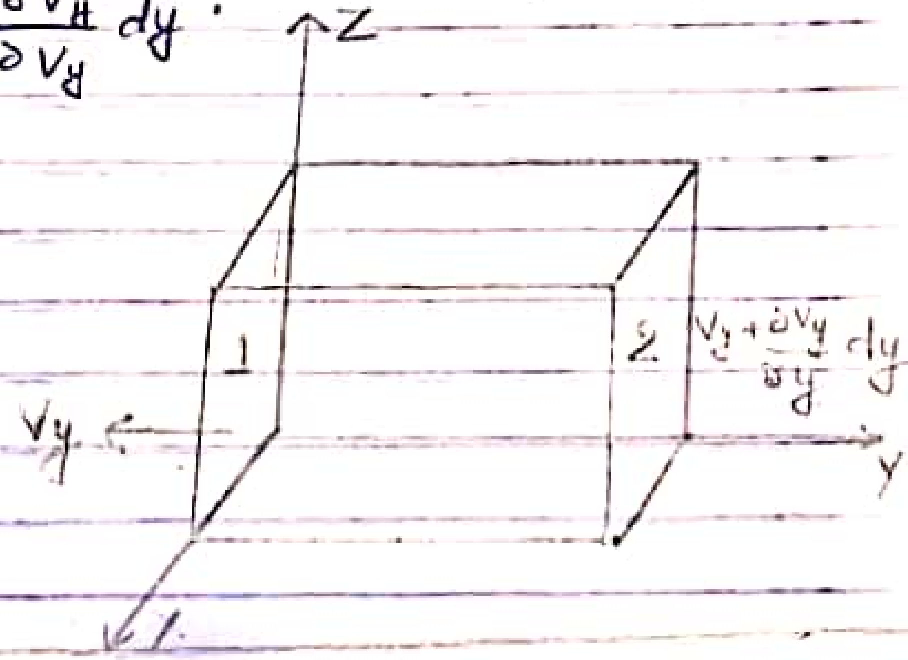
Dr. Supriya Kumari
Deptt. of Physics
J.N College, Madhubani

Vector Analysis (Continued)

Physical significance of Divergence:

Let us consider a small volume $d\tau = dx\,dy\,dz$ and $\vec{V}$ a vector point function at O. Let V_x, V_y, V_z be components of $\vec{V}$ along the x, y, z axes respectively. Similarly, $\frac{\partial V_x}{\partial x}, \frac{\partial V_y}{\partial y}, \frac{\partial V_z}{\partial z}$ be the rates of change of these components in their own components in their own directions.

Surfaces 1 & 2 are perpendicular to the y-axis. The value of Y-components of $\vec{V}$ over this surfaces are $\vec{V_y}$ & $\vec{V_y} + \frac{\partial V_H}{\partial V_y} dy$.



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These values can be assumed to remain constant over the respective surfaces as they are very small.

Vector $\vec{V}$ can be described by any physical meaning. For instance, let $\vec{V} = \rho v$, where ρ is the density and $\vec{v}$ is the velocity of fluid. Then, $\vec{V} = \rho\vec{v}$ represents the rate at which the mass of fluid that flows across a unit area normal to $\vec{v}$ around the point. This flow is very often called the flux of vector $\vec{V}$.

Hence, the net out flow or the flux of vector $\vec{V}$ in the direction is given by,

$$\left(V_y + \frac{\partial V_y}{\partial y} dy\right) dx\, dz - V_y\, dx\, dz$$

$$= \frac{\partial V_y}{\partial y} dx\, dy\, dz.$$

Similarly, the net outflow or the flux of vector $\vec{V}$ in x, z directions are given by $\frac{\partial V_x}{\partial x} dx\, dy\, dz$ & $\frac{\partial V_z}{\partial z} dx\, dy\, dz$ respectively.

Hence, total outflow or the flux of vector $\vec{V}$ from the volume element,

$$= \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right) dx\, dy\, dz$$

$$= \frac{\partial V_y}{\partial y}\, dx\, dy\, dz.$$

Thus, the outward flow of vector $\vec{V}$ per unit volume.

$$= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}.$$

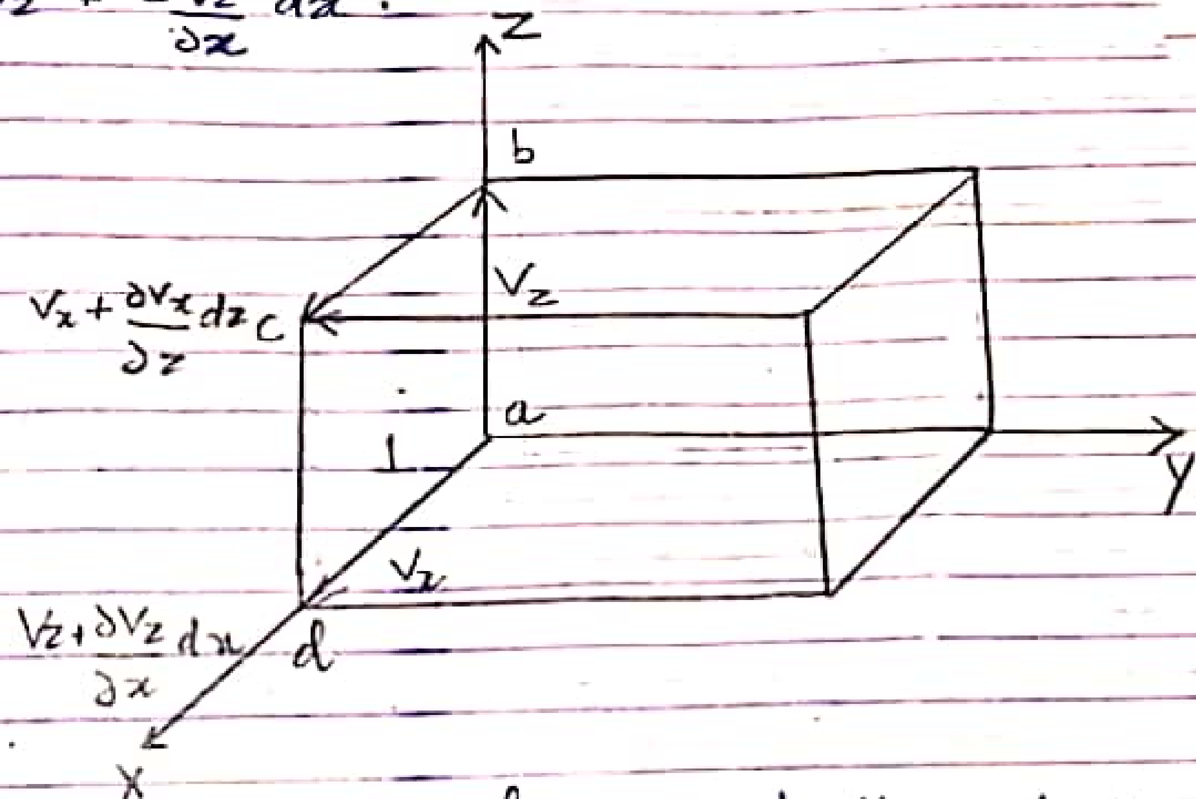
or, $\text{div}\,\vec{V} = \nabla \cdot V = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$.

<u>Curl of a Vector</u> :– If $\vec{V}(x, y, z)$ is a differentiable vector field then the curl $\vec{V}$ or rot $\vec{V}$ is defined by

$$\vec{\nabla} \times \vec{V} \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{K}\right) \times \left(V_1\vec{i} + V_2\vec{j} + V_3\vec{K}\right)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

<u>Physical signifiance :-</u> Let us consider a rectangular path which encloses surface market 1 perpendicular to y-component of $\vec{V}$. The values of the x & z component of the vector $\vec{V}$ along ad, bc, ab and dc are V_x, $V_x + \frac{\partial V_x}{\partial x} dz$; V_z, $V_z + \frac{\partial V_z}{\partial x} dx$.



Hence, the line integral around the contour abcda is

$$V_z d_z + \left(V_x + \frac{\partial V_x}{\partial z} dz\right) dx - \left(V_z + \frac{\partial V_z}{\partial x} dx\right) dz - V_x dx$$

$$= \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) dx\,dz.$$

The contribution towards line integral along cd & da are negative since these paths are described in a sense opposite to the vector concerned. This line integral is maximum since the paths ab, bc, cd & da are taken either parallel or antiparallel to the vectors. Since the value of this maximum line integral per unit area is a vector along the positive normal to the area, we can write

$$\text{Curl}_y \vec{V} = \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)$$

Similarly, $\text{Curl}_x \vec{V} = \left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial y}\right)$

and, $\text{Curl}_z \vec{V} = \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$

Adding we get,

$$\text{Curl}\,\vec{V} = \vec{i}\,\text{curl}_x V + \vec{j}\,\text{Curl}_y V + \vec{k}\,\text{curl}_z V$$

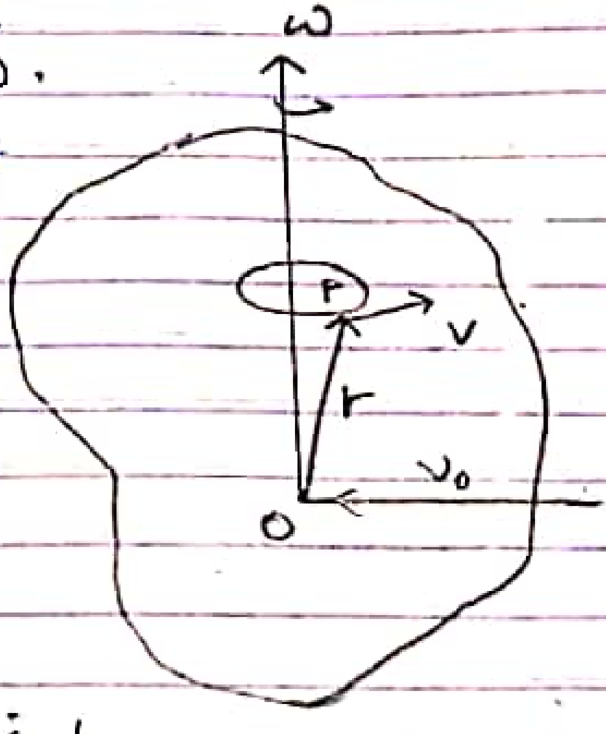
$$= \vec{i}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + j\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \vec{K}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

Curl of a vector :- angular velocity

Let us consider a rigid body rotating about an axis rotating about an axis passing through point O.
Let it rotate with a const. angular velocity ω.
Any point P on the body such that OP = r.
The linear velocity

$V = \omega \times r$



If $\vec{\omega} = \omega_1 \vec{i} + \omega_2 \vec{j} + \omega_3 \vec{k}$;

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

Then,

$$\vec{\omega} \times \vec{r} = \begin{vmatrix} i & j & k \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

or, $\vec{V} = (\omega_2 z - \omega_3 y)\vec{i} + (\omega_3 x - \omega_1 z)\vec{j} + (\omega_1 y - \omega_2 x)\vec{K}$.

$$\therefore \text{Curl } \vec{V} = \left(i\frac{\delta}{\delta V} + j\frac{\delta}{\delta y} + K\frac{\delta}{\delta z}\right) \times \left\{(\omega_2 z - \omega_3 y)\vec{i} + (\omega_3 x - \omega_1 z)\vec{j} + (\omega_1 y - \omega_2 x)\vec{K}\right\}.$$

$$= 2(\omega_1 \vec{i} + \omega_2 \vec{j} + \omega_3 \vec{K}).$$

$$= 2\vec{\omega}.$$

Thus, curl of the linear velocity of a rigid body is twice its angular velocity.

<u>Line integral</u>: Let $r(u) = x(u)\vec{i} + y(u)\vec{j} + z(u)\vec{K}$ is the position vector of (x, y, z) define a curve c joining points P_1 and P_2, where $u = u_1$ and $u = u_2$ respectively.

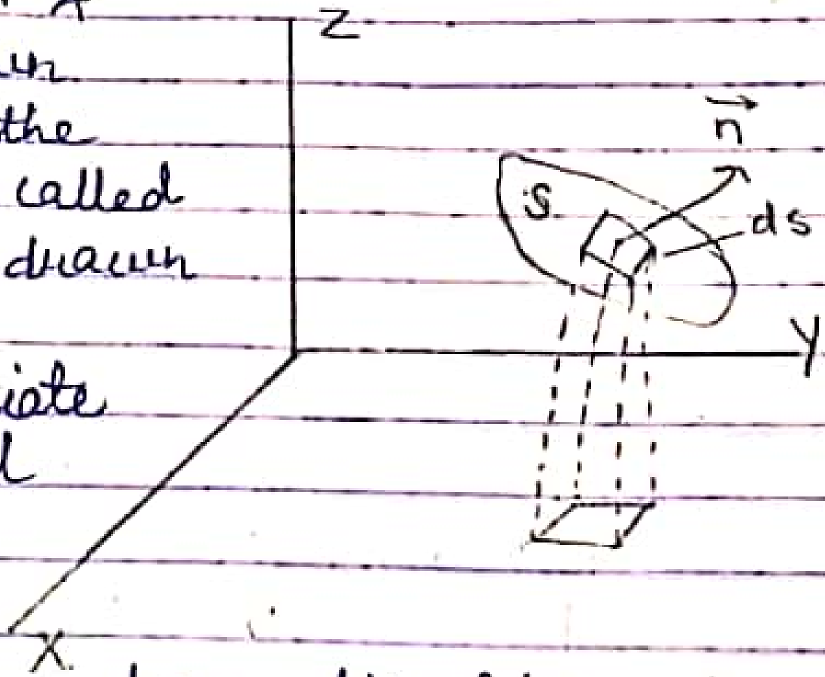
Let us assume that c is composed of a finite number of curves for each of which $r(u)$ has a continuous derivative. Let $A(x, y, z) = A_1\vec{i} + A_2\vec{j} + A_3\vec{K}$ be a vector fun. of position defined and continuous along C from P_1 to P_2 written as,

$$\int_{P_1}^{P_2} \vec{A}\cdot d\vec{r} = \int_C A\cdot dr = \int_C A_1 dx + A_2 dy + A_3 dz.$$

is an example of line integral. In general, any integral which is to be evaluated along a curve is called a line integral.

Surface integrals

Let S be a two sided surface, such as shown in the adjoining figure. Let one side of S be considered arbitrarily as the (+) ve side. A unit normal $\vec{n}$ drawn to any point of the +(ve) side of S is called a +(ve) or outward drawn unit normal.



Let us associate with the differential of surface area, ds a vector $d\vec{s}$ whose magnitude is ds and whose direction is that of $\vec{n}$.

Then, $d\vec{S} = \vec{n} \cdot ds$.
The integral, $$\iint_S \vec{A} \cdot d\vec{s} = \iint_S \vec{A} \cdot \vec{n}\, ds$$
is called the surface integral of $\vec{A}$ over the surface S. Physically, it represents the flux of $\vec{A}$ over S.

<u>Volume integral</u> :- Let us consider a closed surface enclosing a volume V. Then,
$$\iiint_V \vec{A} \cdot dv$$ is called volume integral or space integral of the vector $\vec{A}$, where $\vec{A}$ is a vector point function continuous and defined over the region of volume V.